JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 21, No. 6, November–December 1998

Attitude Stability of DeBra–Delp Satellites in Circular Orbit

M. Temel Aygün* and Umur Daybelge†
Istanbul Technical University, Maslak, 80626 Istanbul, Turkey

DeBra–Delp satellite refers to a particular attitude with respect to the orbital coordinates of certain rigid three-axis satellites. The attitude stability of such a satellite placed on a circular orbit, examples of which can be found among natural and artificial satellites, is examined. As is well known, linearized techniques for the stability of the particular equilibrium of the named satellites have been inconclusive because no Lyapunov function for them was found. Our aim is to demarcate a region in the parameter space (T_1, T_2) defined by the moments of inertia, where the equilibrium of DeBra-Delp satellites is stable. This aim is achieved using and contrasting both analytical and numerical solutions and analyzing nonlinear characteristics of the problem. The analytical approach used is based on Tkhai's theorem, which represents an extension of the KAM theory and provides a sufficient criterion for stability.

I. Introduction

IT is known that a rigid triaxial satellite on a circular orbit has an equilibrium if one of its principal axes of inertia is normal to the orbital plane and another one is tangential to the orbit, and its orbital angular velocity is identical to its spin velocity. As shown by the works of Delp and DeBra,[1] Modi and Brereton,[2,3] and Beletskii,[4] the linearized analysis indicates that these equilibria are stable for a wide range of principal inertial moments irrespective of which principal axes are chosen to be normal and tangential to the orbit (see Fig. 1). The equilibrium, which is called the Lagrange configuration and which Newton intuitively assumed to be valid for the moon, possesses a Lyapunov function that is its Hamiltonian and, thus, its stability is guaranteed. At present, it is generally and tacitly assumed that all natural satellites have ended up in the Lagrangian equilibrium.[5]

On the other hand, after the publication of Ref. 1, Likins showed that the linearized analysis of stability is inadequate for a particular equilibrium, i.e., the DeBra–Delp configuration in Fig. 1 (Ref. 6). The question of stability for DeBra–Delp satellites has remained thus far unresolved.[7,8] Because of errors in their analysis, as we shall subsequently demonstrate, the attempt by Marandi and Modi[9] in 1989 to illuminate the stability behavior of DeBra–Delp satellites has likewise not succeeded. The difficulty of the stability problem for DeBra–Delp satellites is related to the fundamental problem of how to find a criterion of stability for general Hamiltonian systems. The Kolmogorov-Arnold-Moser (KAM) theory[10] provides such a criterion, but it is only for systems with two degrees of freedom. For such systems, the dimension of the phase space is four, and the energy levels are given by three-dimensional manifolds. Hence, the invariant two-dimensional tori divide each energy level set. Thus, any phase space curve, no matter how complicated, is confined to the gap between two invariant tori of the perturbed system. Hence, the corresponding action variables stay forever near their initial values. When the number of degrees of freedom n is greater than two, then the n-dimensional invariant tori do not divide the $(2n-1)$-dimensional energy level manifold. Instead, they are arranged in it like points on a plane or lines in space. Then, the gaps corresponding to various resonances are connected to one another. The phase curves starting near resonances are in this case not prevented by the invariant tori from going far away. Hence, it cannot be expected that the action variables along such a phase curve remain close to their initial values.[11] Furthermore, it was shown by Arnold[12] that there could exist no algorithm that leads in a finite number of steps to a decision on stability in every dynamical system.

Received Oct. 29, 1996; revision received Dec. 5, 1997; accepted for publication April 15, 1998.

*Assistant Professor, Department of Astronautical Sciences and Technology.

†Professor, Department of Astronautical Sciences and Technology.

Nevertheless, Arnold's theorem did not rule out the possibility for an algorithmic sufficient condition. Indeed, Tkhai found, in 1985, such a sufficient condition for arbitrary Hamiltonian systems in normal form.[13] The normalization of higher-order terms of a Hamiltonian requires care, the algebra becomes rather tedious when one normalizes the Hamiltonian up to the terms of degree four, as required by Tkhai's theorem.

II. Analytical Nonlinear Stability Analysis

This section contains the attitude stability analysis of a rigid satellite in a circular orbit investigated by the Tkhai's theorem[13] by use of the normalized Hamiltonian, as first suggested by Marandi and Modi.[9]

A. Normalized Hamiltonian for DeBra–Delp Satellites

Rodrigues parameters r_i, associated with a simple rotation of a rigid body B in reference frame A, provide a local coordinate system for the satellite about its center of mass. The angular velocity of B in A, expressed in terms of $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$, can be written as[8,10]

$$\boldsymbol{\Omega} = \frac{2}{1+|\boldsymbol{r}|^2}(\dot{\boldsymbol{r}} - \boldsymbol{r}\times\dot{\boldsymbol{r}}) \tag{1}$$

Conversely, if $\boldsymbol{\Omega}$ is known as a function of time, the Rodrigues vector can be found by solving the following differential equation:

$$\dot{\boldsymbol{r}} = \tfrac{1}{2}(\boldsymbol{\Omega} + \boldsymbol{r}\times\boldsymbol{\Omega} + \boldsymbol{r}\boldsymbol{r}\cdot\boldsymbol{\Omega}) \tag{2}$$

If Eq. (2) can also be formulated in terms of matrices $\boldsymbol{\Omega}$, $\boldsymbol{r}$, and $\tilde{\boldsymbol{r}}$, then

$$\dot{\boldsymbol{r}} = \tfrac{1}{2}(\boldsymbol{U} - \tilde{\boldsymbol{r}}^T + \boldsymbol{r}^T\boldsymbol{r})\boldsymbol{\Omega} \tag{3}$$

where $\boldsymbol{U}$ is unit matrix and $\tilde{\boldsymbol{r}}$ is skew-symmetric Rodrigues matrix. Then, it is explicitly found as

$$\dot{\boldsymbol{r}} = \frac{1}{2}\begin{bmatrix} 1+r_1^2 & r_1r_2 - r_3 & r_1r_3 + r_2 \\ r_1r_2 + r_3 & 1 + r_2^2 & r_2r_3 - r_1 \\ r_1r_3 - r_2 & r_1 + r_2r_3 & 1 + r_3^2 \end{bmatrix}\boldsymbol{\Omega} \tag{4}$$

The corresponding conjugate variables s_i are found in terms of the angular velocity of the satellite. By considering L as a function of $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$, i.e., $L(\boldsymbol{r}, \dot{\boldsymbol{r}})$, $\boldsymbol{s} = \partial L/\partial\dot{\boldsymbol{r}}$ are computed after the appropriate substitution for $\boldsymbol{\Omega}$ and $\boldsymbol{C}_{ij}$, components of the direction cosine matrix, as

$$\boldsymbol{s} = \frac{2}{1+|\boldsymbol{r}|^2}\begin{bmatrix} I_1 & -r_3I_2 & r_2I_3 \\ r_3I_1 & I_2 & -r_1I_3 \\ -r_2I_1 & r_1I_2 & I_3 \end{bmatrix}(\boldsymbol{\Omega}_i + \boldsymbol{C}_{3i}) \tag{5}$$

where the principal moments of inertia are denoted by I_i. Therefore,

$$\boldsymbol{\Omega} = \boldsymbol{M}\boldsymbol{s} - \boldsymbol{C}_{3i} \tag{6}$$

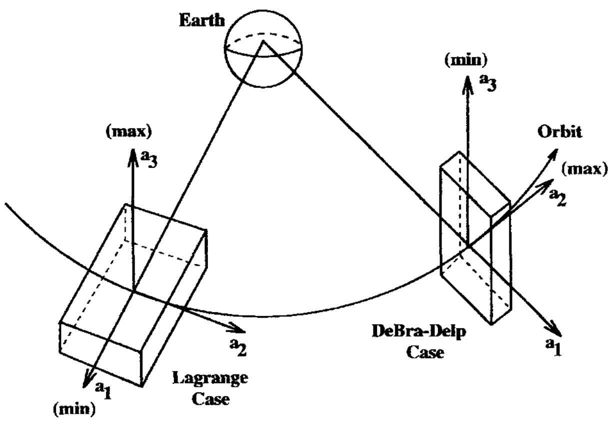

Fig. 1 Configuration of the satellites in the orbit.

where

$$
M = \begin{bmatrix} \dfrac{1+r_1^2}{2I_1} & \dfrac{r_1r_2+r_3}{2I_1} & \dfrac{r_1r_3-r_2}{2I_1} \\ \dfrac{r_1r_2-r_3}{2I_2} & \dfrac{1+r_2^2}{2I_2} & \dfrac{r_1+r_2r_3}{2I_2} \\ \dfrac{r_1r_3-r_2}{2I_3} & \dfrac{-r_1+r_2r_3}{2I_3} & \dfrac{1+r_3^2}{2I_3} \end{bmatrix}
$$

To investigate the attitude stability of a rigid satellite in a circular orbit about the equilibrium, its Hamiltonian of the rotational motion is to be written in the Birkhoff[14] normal form. To achieve this, we first consider the equilibrium for $(\boldsymbol{C}_{ij}, \boldsymbol{\Omega})$ in $(\boldsymbol{r}, \boldsymbol{s})$ coordinates, i.e.,

$$
(\boldsymbol{r}, \boldsymbol{s})[(\boldsymbol{C}_{ij}, \boldsymbol{\Omega})]|_{\text{eq}} = (0, 0, 0, 0, 0, 2I_3)
$$

and define the following canonical transformation $\boldsymbol{r} = \bar{\boldsymbol{r}}$ and $\boldsymbol{s}^T = (\bar{s}_1, \bar{s}_2, \bar{s}_3 + 2I_3)$ to bring the equilibrium to the origin. Then, the Hamiltonian of the attitude motion of a rigid satellite in a circular orbit under the influence of the gravitational torques can be written by a further substitution in terms of $\bar{\boldsymbol{r}}$ and $\bar{\boldsymbol{s}}$:

$$
\begin{aligned}
\mathcal{H}[\boldsymbol{C}_{ij}, \boldsymbol{\Omega}] &= \frac{1}{2}\sum_{i=1}^{3} I_i\left(\Omega_i^2 - C_{3i}^2 + 3C_{1i}^2\right) \\
&= \frac{1}{2}\left(\boldsymbol{\Omega}^T \boldsymbol{I}\boldsymbol{\Omega} - \boldsymbol{C}_{3i}^T \boldsymbol{I}\boldsymbol{C}_{3i} + 3\boldsymbol{C}_{1i}^T \boldsymbol{I}\boldsymbol{C}_{1i}\right) \\
&= \frac{1}{2}\left[\boldsymbol{s}^T \boldsymbol{M}^T \boldsymbol{I}\boldsymbol{M}\boldsymbol{s} - 2\boldsymbol{s}^T \boldsymbol{M}^T \boldsymbol{I}\boldsymbol{C}_{3i} + 3\boldsymbol{C}_{1i}^T \boldsymbol{I}\boldsymbol{C}_{1i}\right] \qquad (7)
\end{aligned}
$$

where the new variables $(\bar{\boldsymbol{r}}, \bar{\boldsymbol{s}})$ are renamed $(\boldsymbol{r}, \boldsymbol{s})$ in the following sections because the original variables are no longer needed.

Thus, the Hamiltonian of the system can be written in the following form:

$$
\mathcal{H} = \tfrac{1}{2}(h_1 + h_2 + h_3) \qquad (8)
$$

where $h_1 = \boldsymbol{s}^T \boldsymbol{M}^T \boldsymbol{I}\boldsymbol{M}\boldsymbol{s}$, $h_2 = -2\boldsymbol{s}^T \boldsymbol{M}^T \boldsymbol{I}\boldsymbol{C}_{3i}$, and $h_3 = 3\boldsymbol{C}_{1i}^T \boldsymbol{I}\boldsymbol{C}_{1i}$ are found explicitly by REDUCE.[15]

Then, the Hamiltonian can be expanded, as described in Ref. 16, depending on the powers of $(\boldsymbol{r}, \boldsymbol{s})$ explicitly in the following form:

$$
\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4 + \cdots \qquad (9)
$$

where

$$
\mathcal{H}_0 = \frac{3I_1 - I_3}{2}, \qquad \mathcal{H}_1 = 0
$$

$$
\begin{aligned}
\mathcal{H}_2 = \frac{1}{2}\Bigg[&\frac{I_3^2}{I_2}r_1^2 + \left(\frac{I_3^2}{I_1} - 12(I_1 - I_3)\right)r_2^2 \\
&+ 12(I_2 - I_1)r_3^2 + \frac{1}{4I_1}s_1^2 + \frac{1}{4I_2}s_2^2 + \frac{1}{4I_3}s_3^2 \\
&+ 2\left(\frac{I_3}{2I_2} - 1\right)s_2r_1 + 2\left(1 - \frac{I_3}{2I_1}\right)s_1r_2\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{H}_3 = {}& \frac{I_3}{2I_2}r_1^2s_3 + \frac{I_3}{2I_1}r_2^2s_3 + \frac{1}{2}r_3^2s_3 \\
&+ \frac{-12I_1I_2^2 + 12I_1I_2I_3 + I_1I_3^2 - I_2I_3^2}{I_1I_2}r_1r_2r_3 \\
&+ \frac{I_3(I_1 - I_2)}{2I_1I_2}r_2r_3s_2 + \frac{I_3(-I_1 + I_2)}{2I_1I_2}r_1r_3s_1 \\
&+ \frac{I_3 - I_2}{4I_2I_3}r_1s_2s_3 + \frac{I_1 - I_3}{4I_1I_3}r_2s_1s_3 + \frac{I_2 - I_1}{4I_1I_2}r_3s_1s_2
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{H}_4 = {}& \frac{1}{4I_1}r_1^2s_1^2 + \frac{1}{8I_3}r_1^2s_2^2 + \frac{1}{8I_2}r_1^2s_3^2 + \frac{1}{8I_3}r_2^2s_1^2 \\
&+ \frac{1}{4I_2}r_2^2s_2^2 + \frac{1}{8I_1}r_2^2s_3^2 + \frac{1}{8I_2}r_3^2s_1^2 + \frac{1}{8I_1}r_3^2s_2^2 \\
&+ \frac{1}{4I_3}r_3^2s_3^2 + \frac{48I_1I_2 - 24I_2^2 - 24I_2I_3 + I_3^2}{2I_2}r_2^2r_3^2 \\
&+ \frac{12I_1^2 - 24I_1I_2 + 12I_1I_3 + I_3^2}{2I_1}r_1^2r_3^2 \\
&+ 6(I_1 + I_2 - 2I_3)r_1^2r_2^2 + 12(I_1 - I_3)r_2^4 \\
&+ \frac{24I_1 - 24I_2 + I_3}{2}r_3^4 + \frac{I_3(I_1 - I_2)}{2I_1I_2}r_1^2r_2s_1 \\
&+ \frac{I_2 - I_3}{2I_2}r_2r_3^2s_1 + \frac{I_3 - I_1}{2I_1}r_1r_3^2s_2 + \frac{I_3(I_1 - I_2)}{2I_1I_2}r_1r_2^2s_2 \\
&+ \frac{-I_1I_2 + I_1I_3 + I_2I_3}{4I_1I_2I_3}r_1r_2s_1s_2 + \frac{I_1I_2 - I_1I_3 + I_2I_3}{4I_1I_2I_3}r_1r_3s_1s_3 \\
&+ \frac{I_1I_2 + I_1I_3 - I_2I_3}{4I_1I_2I_3}r_2r_3s_2s_3 + \frac{I_3(I_1 - I_2)}{I_1I_2}r_1r_2r_3s_3
\end{aligned}
$$

Note that expressions for $\mathcal{H}_3$ and $\mathcal{H}_4$ differ from the erroneous counterparts given in Ref. 9. Because the constant term $\mathcal{H}_0$ plays no role in the analysis it will be omitted. The absence of the first-order terms ($\mathcal{H}_1 = 0$) confirms that $(\boldsymbol{r}, \boldsymbol{s}) = (\boldsymbol{0}, \boldsymbol{0})$ is an equilibrium. The expressions $\mathcal{H}_2, \mathcal{H}_3, \ldots,$ are not invariantly defined, and they are functions of $(\boldsymbol{r}, \boldsymbol{s})$.

Details of the normal form analysis for the attitude motion of a rigid satellite is given in the Appendix.

Consider the problem of Lyapunov stability of the zeroth solution of the canonical system with an analytic Hamiltonian function $\mathcal{H}(\boldsymbol{q}, \boldsymbol{p}, t)$, periodic in time or not explicitly dependent on time, as given in the following form:

$$
\mathcal{H} = \mathcal{H}^0 + \mathcal{H}^1(\boldsymbol{q}, \boldsymbol{p}, t) \qquad (10)
$$

where $\mathcal{H}^0(\tau) = \sum \omega_i\tau_i + \frac{1}{2}\sum \omega_{ij}\tau_i\tau_j$ where $\tau_i = (p_i + q_i)/2$ and $\mathcal{H}^1(\boldsymbol{q}, \boldsymbol{p}, t)$ contains terms of order higher than four in the distance from the equilibrium position, with $\mathcal{H}(0, 0, t) = 0$.

We will assume that the linearized system is stable, all of the characteristic frequencies of the system are purely imaginary and distinct, and the system has no third- and fourth-order resonance relations. We shall investigate the autonomous problem when the Hamiltonian $\mathcal{H}$ is not a function of fixed sign. When it is a function of fixed sign, the stability problem is solved by the Lagrange–Dirichlet theorem.[11,17] With these assumptions, the Hamiltonian of the system can be reduced to the Birkhoff normal form given in Eq. (10).

Theorem 1 (Ref. 13): The periodic system, given in Eq. (10), is Lyapunov stable if the quadratic form of $\bar{\mathcal{H}}_4$ (see Appendix) with fixed sign is in the positive cone, i.e., $\tau_i \geq 0$. It follows from the proof that the condition of fixed sign of the quadratic form given in Eq. (10) ensures the Lyapunov stability along the whole numerical axis $-\infty < t < +\infty$, which is called permanent stability.[14]

Obviously, the quadratic form of $\bar{\mathcal{H}}_4$ with fixed sign also ensures the stability of an autonomous system. However, the result can be amplified in that case.

Theorem 2 (Ref. 13): The autonomous system given by Eq. (10) is Lyapunov stable if the system of equations

$$\sum_{i=1}^{n} \omega_i \tau_i = 0 \tag{11}$$

$$\sum_{i=1}^{n} \omega_{ij} \tau_i \tau_j = 0 \tag{12}$$

has no solution other than trivial one, $\forall \tau_i \geq 0$ and $\exists \boldsymbol{\tau} \neq 0$.

The application of theorem 2, to the investigation of the stability of the steady motion of a mechanical system with cyclic (ignorable) coordinates, may ensure absolute stability, as shown in the next subsection.

B. Application of Tkhai's Theorem to the DeBra–Delp Satellite

Clearly, the sufficient condition for Lyapunov stability is fulfilled when numbers in at least one of the sets ω_i and ω_{ij} $(i, j = 1, 2, 3)$ are of the same sign. This can be determined by a simple inspection of signs once these numbers are determined. In the Lagrange region, ω_i are positive definite. Hence, this region is Lyapunov stable. In the DeBra–Delp region, however, ω_1 is negative, whereas ω_2 and ω_3 are both positive. Similarly, the coefficients ω_{ij} have likewise different signs: ω_{11}, ω_{12}, and ω_{22} are always negative, whereas ω_{13}, ω_{23}, and ω_{33} have different signs.[16] It is very important to note that ω_{ij} take the highest and the lowest values, generally, out of the DeBra–Delp region or near to or on the resonance curves of degree three or four.

If this is the case, the plane and the cone (Fig. 2), expressed by these equations in the space τ_1, τ_2, and τ_3, have an intersection in the first quadrant passing through the origin. In this case, according to the Tkhai's theorem, we can say nothing about the stability. Going from point to point in the DeBra–Delp region of parameters (T_1, T_2), which are defined as $T_1 = (I_2 - I_3)/I_1$ and $T_2 = (I_3 - I_1)/I_2$, we can thus demarcate some region of stability from regions of undecided nature. Such a demarcation over the DeBra–Delp region between stable and unknown subregions can be done numerically using a grid of relevant mesh size. However, even the unknown subregions may still contain a subset where the equilibrium is stable. Unfortunately, the demarcation given in Ref. 9 is incorrect due to several errors in $\mathcal{H}_3$ and $\mathcal{H}_4$. Besides, their numerical demarcation criterion is unnecessarily complicated. We indicate next a straightforward method for this purpose.

Equation (11) can be written explicitly for $n = 3$ as

$$\omega_1 \tau_1 + \omega_2 \tau_2 + \omega_3 \tau_3 = 0 \tag{13}$$

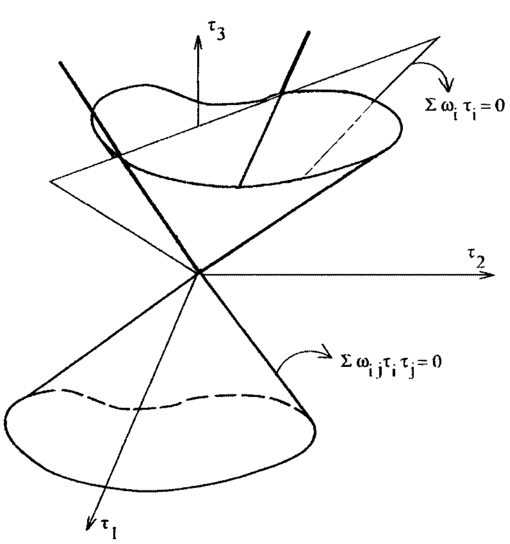

Fig. 2 Plane and cone intersections in τ_i.

Because ω_1 is negative and ω_2 and ω_3 are both positive in the DeBra–Delp region, we can write Eq. (13) as

$$|\omega_1| \tau_1 = \omega_2 \tau_2 + \omega_3 \tau_3 \tag{14}$$

According to the definitions of theorem 1 for τ_i, which are always positive and at least one of them is other than zero, they can be defined in the spherical coordinates, assuming $|\boldsymbol{\tau}| = 1$, as follows:

$$\tau_1 = \sin\theta \cos\varphi \tag{15}$$

$$\tau_2 = \sin\theta \sin\varphi \tag{16}$$

$$\tau_3 = \cos\theta \tag{17}$$

Then, Eq. (14) is found as

$$|\omega_1| \sin\theta \cos\varphi = \omega_2 \sin\theta \cos\varphi + \omega_3 \cos\theta \tag{18}$$

and dividing by $\cos\theta$, ω_3 is found as

$$\omega_3 = \tan\theta (|\omega_1| \cos\varphi - \omega_2 \sin\varphi) \tag{19}$$

Thus,

$$\tan\theta = \frac{\omega_3}{|\omega_1| \cos\varphi - \omega_2 \sin\varphi} \tag{20}$$

Because, $\tau_i > 0$, then $0 < \varphi < \pi/2$ and $0 < \theta < \pi$ must be satisfied. As a result, $\tan\theta$ must satisfy the following equation:

$$\tan\theta = \frac{-\omega_3}{\omega_1 \cos\varphi + \omega_2 \sin\varphi} \tag{21}$$

By taking the second condition, which is given by Eq. (12), and by using the definition in Eqs. (15–17), it can be written as

$$\begin{aligned} &\omega_{11} \sin^2\theta \cos^2\varphi + \omega_{22} \sin^2\theta \sin^2\varphi + \omega_{33} \cos^2\theta \\ &\quad + 2\omega_{12} \sin^2\theta \cos\varphi \sin\varphi + 2\omega_{13} \sin\theta \cos\theta \cos\varphi \\ &\quad + 2\omega_{23} \sin\theta \cos\theta \sin\varphi = 0 \end{aligned} \tag{22}$$

After some algebra, Eq. (22) is reduced to

$$\mathcal{A} \tan^2\theta + 2\mathcal{B} \tan\theta + \mathcal{C} = 0 \tag{23}$$

where

$$\mathcal{A} = \omega_{11} \cos^2\varphi + \omega_{22} \sin^2\varphi + \omega_{12} \sin 2\varphi$$

$$\mathcal{B} = \omega_{13} \cos\varphi + \omega_{23} \sin\varphi, \qquad \mathcal{C} = \omega_{33}$$

Thus,

$$\tan\theta_{1,2} = (-\mathcal{B} \mp \mathcal{D})/\mathcal{A} \tag{24}$$

where

$$\mathcal{D} = \sqrt{\mathcal{B}^2 - \mathcal{AC}}$$

Equations (21) and (24) yield a common solution for θ if there is a solution for τ in Eqs. (11) and (12) other than the trivial one. As a check, we can run a program by choosing φ with an appropriate increment, calculating θ according to the value of chosen φ, and then comparing the solutions as to whether they have same value with respect to the parameters T_1 and T_2 in the DeBra–Delp region.

The result of the application of Tkhai's theorem to the DeBra–Delp satellites is given in Fig. 3. In this figure, dots $(\cdot)$ represent the stability region, circles $(\circ)$ represent the unknown region where there are other solutions than the trivial one, and pluses $(+)$ represent a very narrow region, or rather line, where the sum of absolute values of ω_{ij} is greater or equal to the value of 100,000, i.e.,

$$\sum_{i,j=1}^{3} |\omega_{ij}| \geq 100{,}000$$

One of the important results to be extracted from Fig. 3 is that there is a well-defined attitude stability region for the DeBra–Delp satellites. Another important result is that the lines denoted by $+$ coincide with the resonance curves of degree of three and four. Thus, the presence of resonance curves in the parameter space has manifested itself by such numerical singularities in the computation. We also note that there is almost no problem for the attitude stability

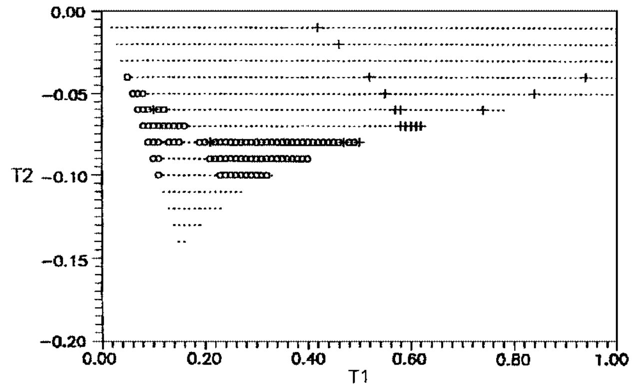

Fig. 3 Stability regions for the attitude motion of DeBra–Delp satellites.

up to the value of $T_2 = -0.07$ in the DeBra–Delp region. This value may be raised to $T_2 = -0.05$ by placing T_1 parameters in the range between 0 and 1. Thus, the stability region shown in Fig. 3 differs from the incorrect one given in Ref. 9.

III. Conclusion

This study has dealt with the long standing problem of the attitude stability of a rigid DeBra–Delp satellite under the influence of the gravitational torques in a circular orbit. Marandi and Modi[9] claimed to have made a breakthrough in 1989 for the determination of the stability of these satellites by using Tkhai's sufficiency criteria.[13] We have shown, however, that their results were incorrect because of the algebraic errors in some components $\mathcal{H}_3$ and $\mathcal{H}_4$.

We have given a complete derivation of the Hamiltonian problem of attitude motion for rigid satellites in a circular orbit with arbitrary orientation. The linear theory of stability, however, is shown to be inconclusive for a particular initial configuration, which is called DeBra–Delp case because of indefinite sign of the Hamiltonian. Therefore, the attitude stability in the DeBra–Delp case has to be analyzed by the nonlinear theory.

We have applied Tkhai's theorem for an analytical approach to obtain a demarcation in parameter space for securely stable regions similar to Marandi and Modi, but correcting the algebraic errors. For this purpose, we have made use of the REDUCE symbolic programming language to manipulate extensive algebraic operations and expressions to avoid some errors of the former analysis. Tkhai's criteria were then applied using a much simpler geometrical test than used in Ref. 9, yielding a fast numerical decision. This analytical approach gave us stable region over the whole parameter space.

These conclusions were also checked and verified by independent approximative and asymptotic methods used in our related work (see Ref. 16) based on Lyapunov exponents and phase space analysis.

Appendix: Normal Form Analysis

A. Normalization of $\mathcal{H}_2$

After constructing the Hamiltonian of the system up to fourth degree as polynomials, to analyze the stability of the system, the sufficient condition for canonical transformation of the system to normal form is that the eigenvalues (characteristic frequencies) of the linear system ($\boldsymbol{L}$) must be pure imaginary and distinct.[10,12,17]

1. *Linear System Analysis*

The dynamical system associated with $\mathcal{H}$ is defined

$$\begin{pmatrix}\dot{\boldsymbol{r}}\\ \dot{\boldsymbol{s}}\end{pmatrix} = \boldsymbol{J}\begin{pmatrix}\dfrac{\partial\mathcal{H}}{\partial\boldsymbol{r}}\\[2mm] \dfrac{\partial\mathcal{H}}{\partial\boldsymbol{s}}\end{pmatrix}$$

which is obviously set of nonlinear equations, where $\boldsymbol{J}$ is $2n \times 2n$ skew-symmetric unity matrix.

The linear part of this system comes from $\mathcal{H}_2$ as follows:

$$\begin{pmatrix}\dot{\boldsymbol{r}}\\ \dot{\boldsymbol{s}}\end{pmatrix} = \boldsymbol{L}\begin{pmatrix}\boldsymbol{r}\\ \boldsymbol{s}\end{pmatrix}$$

where

$$\boldsymbol{L} = \boldsymbol{J}\left[\frac{\partial^2\mathcal{H}_2}{\partial(\boldsymbol{r},\boldsymbol{s})^2}\right]$$

whose components, which are other than zero, can be written as

$$L_{12} = -L_{53} = \frac{2I_1 - I_3}{2I_1}, \qquad L_{13} = \frac{1}{4I_1}$$

$$L_{21} = -L_{42} = \frac{-2I_2 + I_3}{2I_2}, \qquad L_{24} = \frac{1}{4I_2}$$

$$L_{36} = \frac{1}{4I_3}, \qquad L_{41} = -\frac{I_3^2}{I_2}$$

$$L_{52} = \frac{12I_1(I_1 - I_3) - I_1 I_3^2}{I_1}, \qquad L_{65} = \frac{12I_3(I - 1 - I_2)}{I_3}$$

If the eigenvalues of linear system matrix $\boldsymbol{L}$ are distinct and pure imaginary, then by a linear canonical transformation $\boldsymbol{T}$ via the symplectic basis[10,17] $(\boldsymbol{r},\boldsymbol{s})$ variables transform to $(\boldsymbol{x},\boldsymbol{y})$ variables by

$$\begin{pmatrix}\boldsymbol{r}\\ \boldsymbol{s}\end{pmatrix} = \boldsymbol{T}\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}$$

Then, the system equations of motion can be written as

$$\begin{pmatrix}\dot{\boldsymbol{x}}\\ \dot{\boldsymbol{y}}\end{pmatrix} = \boldsymbol{T}^{-1}\boldsymbol{L}\boldsymbol{T}\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}$$

where $\boldsymbol{L}$ is in the canonical form of

$$\boldsymbol{T}^{-1}\boldsymbol{L}\boldsymbol{T} = \begin{bmatrix}\boldsymbol{0} & \boldsymbol{W}\\ -\boldsymbol{W} & \boldsymbol{0}\end{bmatrix}$$

and $\boldsymbol{W} = \mathrm{diag}(\omega_1, \omega_2, \omega_3)$, where ω_i are real and defined as $\omega_1 = \pm\lambda_1$, $\omega_2 = \pm\lambda_2$, and $\omega_3 = \pm\lambda_3$.

It is seen from matrix $\boldsymbol{L}$ that the conjugate variables r_3 and s_3 are decoupled from the rest. Thus, the whole linear system can be thought as two different linear systems. The characteristic equation of the first system, i.e., the equation resulting with the eigenvalues of the first system, is found from the equation $|\lambda_i\boldsymbol{U} - \boldsymbol{L}_1| = 0$ $(i = 1, 2)$ as $\lambda^4 + B\lambda^2 + C = 0$. Thus, the eigenvalues $(\pm j\lambda_1, \pm j\lambda_2)$ of the first system are found as $\lambda_1 = \pm\sqrt{[(B - \sqrt{D})/2]}$ and $\lambda_2 = \pm\sqrt{[(B + \sqrt{D})/2]}$, where λ_1 and λ_2 are real and are assumed $0 \leq \lambda_1 \leq \lambda_2$ (Refs. 9 and 16).

When similar steps are taken for the second linear system equation, the characteristic equation of the second system and eigenvalues $(\pm j\lambda_3)$ are found as $\lambda_3 = \pm\sqrt{(-3T_3)}$.

To make a linear canonical transformation, not only the coefficients of characteristic equation must be real and positive ($B \geq 0$ and $C \geq 0$), but the discriminant of it must also be real ($D \geq 0$). Besides, adding the condition $T_1 + T_2 \geq 0$, from the third eigenvalue relation, we may define a region in the T_1T_2 parameter space. This region characterizes all those Hamiltonian differential equations whose linearization at the considered equilibrium has only pure imaginary eigenvalues.

Definition (Refs. 10, 11, 17): The characteristic frequencies ω_1, ω_2, and ω_3 satisfy a resonance relation of order M if there exist integers m_i not all equal to zero such that

$$\sum_{i=1}^{3} m_i\omega_i = 0$$

with

$$\sum_{i=1}^{3} |m_i| = M \geq 0$$

The resonance curves in the DeBra–Delp region satisfying the condition of $M \leq 4$ are shown in Refs. 9 and 16. Outside these resonance curves the normalization procedure up to the fourth degree is possible.

2. Finding Symplectic Basis

The eigenvectors of the first system, $\boldsymbol{L}_1$, are found for $(i = 1, 2)$ as

$$\boldsymbol{e}_i = \begin{bmatrix} k_i \dfrac{(1+T_1)}{2I_2\left(2s_i^2 - T_1T_2 + 1\right)} \\ k_i \dfrac{\left(s_i^2 - T_1\right)}{2I_2 s_i\left(2s_i^2 - T_1T_2 + 1\right)} \\ k_i \dfrac{\left[(-I_1 - I_2T_2 + 2I_2)s_i^2 - I_1 - I_2T_1T_2 - I_2T_2 + I_2\right]}{I_2 s_i\left(2s_i^2 - T_1T_2 + 1\right)} \\ k_i \end{bmatrix}$$

$$k_i^2 = \left[\frac{\omega_i\left(I_2I_3 + 2I_1I_2\omega_i^2 - 2I_1I_2 - I_3^2 + I_1I_3\right)^2}{I_2^2I_3 + 2I_2^2I_1\omega_i^2 - 2I_1I_2^2 - 2I_2I_3^2 - 2I_1I_2I_3\omega_i^2 + 4I_1I_2I_3 + I_1^2I_2\omega_i^4 - I_2I_1^2 + I_3^3 - 2I_3^2I_1 + I_3I_1^2}\right]$$

where s_i are pure imaginary roots of the first system. Note that first and fourth components of $\boldsymbol{e}_i$ are real, and second and third components of $\boldsymbol{e}_i$ are pure imaginary, assuming k_i as real constant.

Let $\boldsymbol{e}_1$ be the eigenvector corresponding to the eigenvalue $j\lambda_1$. Define $\epsilon_1 N_1^2 = \mathrm{Im}(\boldsymbol{e}_1)\,\boldsymbol{J}\,\mathrm{Re}(\boldsymbol{e}_1)$, where $|\epsilon_1| = 1$ ve $N_1 \geq 0$. It follows that

$$\epsilon_1 N_1^2 = \begin{pmatrix} 0 \\ \bar{e}_{21} \\ \bar{e}_{31} \\ 0 \end{pmatrix}^T \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{pmatrix} e_{11} \\ 0 \\ 0 \\ e_{41} \end{pmatrix}$$

and

$$\epsilon_1 N_1^2 = \begin{bmatrix} 0 & \bar{e}_{21} & \bar{e}_{31} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ e_{41} \\ -e_{11} \\ 0 \end{bmatrix}$$
$$= \bar{e}_{21}e_{41} - e_{11}\bar{e}_{31}$$

where $\bar{e}_{21} = -je_{21}$ and $\bar{e}_{31} = -je_{31}$. As a result, $\boldsymbol{T}_1$ is found in the following form:

$$\boldsymbol{T}_1 = \begin{bmatrix} e_{11} & e_{12} & 0 & 0 \\ 0 & 0 & -\bar{e}_{21} & -\bar{e}_{22} \\ 0 & 0 & -\bar{e}_{31} & -\bar{e}_{32} \\ e_{41} & e_{42} & 0 & 0 \end{bmatrix}$$

Define

$$\boldsymbol{b}_1 = (1/N_1)\,\mathrm{Re}(\boldsymbol{e}_1), \qquad \boldsymbol{b}_4 = (1/\epsilon_1 N_1)\,\mathrm{Im}(\boldsymbol{e}_1)$$

Thus,

$$\boldsymbol{L}_1\boldsymbol{b}_1 = \frac{1}{N_1}\boldsymbol{L}_1\,\mathrm{Re}(\boldsymbol{e}_1) = -\frac{\lambda_1}{N_1}\,\mathrm{Im}(\boldsymbol{e}_1)$$
$$= -\lambda_1\epsilon_1\frac{\mathrm{Im}(\boldsymbol{e}_1)}{\epsilon_1 N_1} = -\lambda_1\epsilon_1\boldsymbol{b}_4$$

and

$$\boldsymbol{L}_1\boldsymbol{b}_4 = (1/N_1)\boldsymbol{L}_1\,\mathrm{Im}(\boldsymbol{e}_1) = (\lambda_1/\epsilon_1 N_1)\,\mathrm{Re}(\boldsymbol{e}_1)$$
$$= \lambda_1\epsilon_1\boldsymbol{b}_1$$

Similarly, define

$$\boldsymbol{b}_2 = (1/N_2)\,\mathrm{Re}(\boldsymbol{e}_2), \qquad \boldsymbol{b}_5 = (1/\epsilon_2 N_2)\,\mathrm{Im}(\boldsymbol{e}_2)$$

Then, in the symplectic basis $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_4, \boldsymbol{b}_5\}$, $\boldsymbol{L}_1$ reads

$$\begin{bmatrix} 0 & 0 & \epsilon_1\lambda_1 & 0 \\ 0 & 0 & 0 & \epsilon_2\lambda_2 \\ -\epsilon_1\lambda_1 & 0 & 0 & 0 \\ 0 & -\epsilon_2\lambda_2 & 0 & 0 \end{bmatrix}$$

By choosing $\epsilon_i = -1$ and $N_i = 1$, constants k_i $(i = 1, 2)$ are found as

The eigenvector of the second system has a form as

$$\boldsymbol{e}_i = \begin{bmatrix} k_i \dfrac{1}{4I_3 s_i} \\ k_i \end{bmatrix}$$

Define $\epsilon_3 N_3^2 = \mathrm{Im}(\boldsymbol{e}_3)\,\boldsymbol{J}\,\mathrm{Re}(\boldsymbol{e}_3)$, i.e., explicitly

$$\epsilon_3 N_3^2 = \begin{pmatrix} \dfrac{-k_3}{4\omega_3 I_3} \\ 0 \end{pmatrix}^T \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ k_3 \end{pmatrix}$$

Then, $\epsilon_3 N_3^2 = k_3^2/4\omega_3 I_3$, where $\omega_3 = \sqrt{(-3T_3)}$ and k_3 is a real constant. Similarly, choosing $\epsilon_3 = -1$ and $N_3 = 1$, k_3 is found as $k_3 = 2\sqrt{(\omega_3 I_3)}$. Therefore,

$$\boldsymbol{T}_2 = \begin{bmatrix} 0 & \frac{1}{2}\sqrt{\omega_3 I_3} \\ 2\sqrt{\omega_3 I_3} & 0 \end{bmatrix}$$

By using the whole symplectic basis, one can construct the linear canonical transformation $(\boldsymbol{r}, \boldsymbol{s}) \rightarrow (\boldsymbol{x}, \boldsymbol{y})$ so that using $\boldsymbol{T}$, where

$$\boldsymbol{T} = \begin{bmatrix} e_{11} & e_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\bar{e}_{21} & -\bar{e}_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\sqrt{\omega_3}I_3 \\ 0 & 0 & 0 & -\bar{e}_{31} & -\bar{e}_{32} & 0 \\ e_{41} & e_{42} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\sqrt{\omega_3}I_3 & 0 & 0 & 0 \end{bmatrix}$$

we can transform $\mathcal{H}_2(\boldsymbol{r}, \boldsymbol{s}) \rightarrow \mathcal{H}_2(\boldsymbol{x}, \boldsymbol{y})$. Thus, the new Hamiltonian now becomes

$$\mathcal{H}_2 = \sum_{i=1}^{3} \omega_i \tau_i$$

where $\tau_i = (x_i^2 + y_i^2/2)$.

B. Normalization of $\mathcal{H}_3$

It is important to note that if characteristic frequencies ω_i do not satisfy any resonance relation of order M three or smaller, then there is a canonical coordinate system in a neighborhood of equilibrium position in which the expression of the Hamiltonian in the new variables is reduced to a Birkhoff normal form of degree three.[10,12,17]

A generating function K can be found, which yields a canonical transformation leading to the elimination of terms of degree three in the new Hamiltonian. By using the following complex linear canonical transformation $\boldsymbol{x} = z/2 + j\boldsymbol{q}$ and $\boldsymbol{y} = jz/2 + \boldsymbol{q}$, $\mathcal{H}_2$ becomes

$$\mathcal{H}_2 = j\sum_{i=1}^{3}\omega_i z_i q_i$$

where $z_i q_i = -j(x_i^2 + y_i^2)/2$. Then, by Jacobi's theorem,[16] the implicit relations $\boldsymbol{q} = \partial K/\partial \boldsymbol{z}$ and $\boldsymbol{u} = \partial K/\partial \boldsymbol{v}$ define a canonical transformation $(\boldsymbol{z}, \boldsymbol{q}) \to (\boldsymbol{u}, \boldsymbol{v})$ about $(z_0, \boldsymbol{q}_0)$, provided the generating function $K(\boldsymbol{v}, \boldsymbol{z})$ satisfies the condition

$$\det\left(\frac{\partial^2 K}{\partial \boldsymbol{v}\partial \boldsymbol{z}}\right)_{v_0, z_0} \neq 0$$

with $\boldsymbol{q}_0 = \partial K/\partial z|_{v_0, z_0}$. Let $K = \boldsymbol{v}\boldsymbol{z} + K_3(\boldsymbol{v}, \boldsymbol{z})$, where K_3 is a homogeneous polynomial of degree three. Then,

$$\det\left(\frac{\partial^2 K}{\partial \boldsymbol{v}\partial \boldsymbol{z}}\right)(0, 0) = \det[\boldsymbol{J}] = 1$$

with $\boldsymbol{q} = \boldsymbol{v} + \partial K_3/\partial \boldsymbol{z}$ and $\boldsymbol{u} = \boldsymbol{z} + \partial K_3/\partial \boldsymbol{v}$. Because K is analytic, the last equality determines $\boldsymbol{z}$ as a unique analytic function of $\boldsymbol{u}$ and $\boldsymbol{v}$, which maps $(0, 0) \to (0)$ (Ref. 9). By substitution of $\boldsymbol{u}$ whenever $\boldsymbol{z}$ appears, $\boldsymbol{z}$ and $\boldsymbol{q}$ become $\boldsymbol{z} = \boldsymbol{u} - \partial K_3/\partial \boldsymbol{v}(\boldsymbol{u}, \boldsymbol{v}) + \mathcal{O}(3)$ and $\boldsymbol{q} = \boldsymbol{v} + \partial K_3/\partial \boldsymbol{u}(\boldsymbol{u}, \boldsymbol{v}) + \mathcal{O}(3)$.

The expansion of $\mathcal{H}$ in terms of the new conjugate variables $(\boldsymbol{u}, \boldsymbol{v})$ up to the terms of degree three $\mathcal{H}_3(\boldsymbol{z}, \boldsymbol{q}) \to \bar{\mathcal{H}}_3(\boldsymbol{u}, \boldsymbol{v})$ is found as

$$\begin{aligned}\mathcal{H}(\boldsymbol{z}, \boldsymbol{q}) &= j\sum_{i=1}^{3}\omega_i\left(u_i - \frac{\partial K_3}{\partial v_i}\right) \\ &\quad\times\left(v_i + \frac{\partial K_3}{\partial u_i}\right) + \mathcal{H}_3(\boldsymbol{u}, \boldsymbol{v}) + \mathcal{O}(4) \\ &= j\sum_{i=1}^{3}\omega_i u_i v_i + j\sum_{i=1}^{3}\omega_i\left(\frac{u_i\partial K_3}{\partial u_i} - \frac{v_i\partial K_3}{\partial v_i}\right) \\ &\quad + \mathcal{H}_3(\boldsymbol{u}, \boldsymbol{v}) + \mathcal{O}(4)\end{aligned}$$

Any term in K_3 may be written as $s_{\boldsymbol{mn}}\boldsymbol{z}^{\boldsymbol{m}}\boldsymbol{v}^{\boldsymbol{n}}$, with $|\boldsymbol{m}| + |\boldsymbol{n}| = 3$, which is an abbreviation for $s_{\boldsymbol{mn}} z_1^{m_1} z_2^{m_2} z_3^{m_3} v_1^{n_1} v_2^{n_2} v_3^{n_3}$, with $|m_1| + |m_2| + |m_3| + |n_1| + |n_2| + |n_3| = 3$. Here, $s_{\boldsymbol{mn}}$ is the coefficient of the $\boldsymbol{z}^{\boldsymbol{m}}\boldsymbol{v}^{\boldsymbol{n}}$ monomial. Hence, the coefficient of $\boldsymbol{u}^{\boldsymbol{m}}\boldsymbol{v}^{\boldsymbol{n}}$ in the third-degree terms of $\bar{\mathcal{H}}$ can be written as

$$\bar{\mathcal{H}}_3 = s_{\boldsymbol{mn}}\left[j\sum_{i=1}^{3}\omega_i(n_i - m_i)\right] + h_{\boldsymbol{mn}}$$

where $h_{\boldsymbol{mn}}$ is the coefficient of the monomial $\boldsymbol{u}^{\boldsymbol{m}}\boldsymbol{v}^{\boldsymbol{n}}$ in $\mathcal{H}_3(\boldsymbol{u}, \boldsymbol{v})$. If there are no resonances of order smaller than or equal to three among ω_i, then $s_{\boldsymbol{mn}}$ may be chosen so that the preceding expression becomes zero. Therefore, an appropriate choice of K eliminates the third degree terms of $\bar{\mathcal{H}}$ (Ref. 16).

C. Normalization of $\mathcal{H}_4$

Now, we have

$$\mathcal{H}(\boldsymbol{u}, \boldsymbol{v}) = j\sum_{i=1}^{3}\omega_i u_i v_i + \bar{\mathcal{H}}_4(\boldsymbol{u}, \boldsymbol{v}) + \mathcal{O}(5)$$

where

$$\bar{\mathcal{H}}_4 = \mathcal{H}_4(\boldsymbol{u}, \boldsymbol{v}) - j\sum_{i=1}^{3}\omega_i\left(\frac{\partial K_3}{\partial \boldsymbol{u}}\cdot\frac{\partial K_3}{\partial \boldsymbol{v}}\right)$$

Similar to the procedure in the preceding subsection, a generating function is found that normalizes $\bar{\mathcal{H}}$ through terms of degree four. Expansion of $\bar{\mathcal{H}}$ through terms of degree four in variables $\boldsymbol{u}$, $\boldsymbol{v}$ is as follows:

$$\bar{\mathcal{H}}(\boldsymbol{u}, \boldsymbol{v}) = j\sum_{i=1}^{3}\omega_i u_i v_i + \bar{\mathcal{H}}_4(\boldsymbol{u}, \boldsymbol{v}) + \mathcal{O}(5)$$

As before, $K_4(\boldsymbol{v}, \boldsymbol{f})$ is a homogeneous polynomial of degree four. The implicit relations are $\boldsymbol{v} = \boldsymbol{f} + \partial K_4/\partial \boldsymbol{u}$ and $\boldsymbol{d} = \boldsymbol{u} + \partial K_4/\partial \boldsymbol{f}$ define an analytic canonical transformation $(\boldsymbol{u}, \boldsymbol{v}) \to (\boldsymbol{d}, \boldsymbol{f})$ close to identity. As before, $\boldsymbol{u} = \boldsymbol{d} - \partial K_4/\partial \boldsymbol{f}(\boldsymbol{u}, \boldsymbol{v}) + \mathcal{O}(4)$, and $\boldsymbol{v} = \boldsymbol{f} + \partial K_4/\partial \boldsymbol{d}(\boldsymbol{u}, \boldsymbol{v}) + \mathcal{O}(4)$. Then,

$$\begin{aligned}\bar{\mathcal{H}}(\boldsymbol{u}, \boldsymbol{v}) &= j\sum_{i=1}^{3}\omega_i\left(d_i - \frac{\partial K_4}{\partial f_i}\right)\left(f_i + \frac{\partial K_4}{\partial d_i}\right) + \mathcal{H}_4(\boldsymbol{d}, \boldsymbol{f}) + \mathcal{O}(5) \\ &= j\sum_{i=1}^{3}\omega_i d_i f_i + j\sum_{i=1}^{3}\omega_i\left(d_i\frac{\partial K_4}{\partial d_i} - f_i\frac{\partial K_f}{\partial f_i}\right) \\ &\quad + j\sum_{i=1}^{3}\omega_i\left(\frac{\partial K_3}{\partial u_i}\cdot\frac{\partial K_3}{\partial v_i}\right)\Bigg|_{(\boldsymbol{d},\boldsymbol{f})} + \mathcal{O}(5)\end{aligned}$$

Any term in K_4 may be written as $b_{\boldsymbol{mn}}\boldsymbol{d}^{\boldsymbol{m}}\boldsymbol{f}^{\boldsymbol{n}}$, with $|\boldsymbol{m}| + |\boldsymbol{n}| = 4$. Here, $s_{\boldsymbol{mn}}$ is the coefficient of the $\boldsymbol{d}^{\boldsymbol{m}}\boldsymbol{f}^{\boldsymbol{n}}$ monomial. Hence, the coefficient of $\boldsymbol{d}^{\boldsymbol{m}}\boldsymbol{f}^{\boldsymbol{n}}$ in fourth-degree terms of $\mathcal{H}_4$ reads as

$$\bar{\mathcal{H}}_4 = s_{\boldsymbol{mn}}\left[j\sum_{i=1}^{3}\omega_i(n_i - m_i)\right] + h_{\boldsymbol{mn}}$$

where $h_{\boldsymbol{mn}}$ is the coefficient of the monomial $\boldsymbol{d}^{\boldsymbol{m}}\boldsymbol{f}^{\boldsymbol{n}}$ in $\mathcal{H}_4(\boldsymbol{d}, \boldsymbol{f})$. The appropriate choice of the coefficients of K_4 gives the Hamiltonian in the normal form, whereas other coefficients are canceled out. Unlike the third degree case, even in the absence of resonance of order four or less, it is not possible to eliminate all the fourth degree terms. For $m_i = n_i$, $(i = 1, 2, 3)$, the monomials $b_{\boldsymbol{m},\boldsymbol{n}}\boldsymbol{d}^{\boldsymbol{m}}\boldsymbol{f}^{\boldsymbol{n}}$ remain. Therefore, K_4 may be chosen such that $\bar{\mathcal{H}}_4$ finally becomes

$$\bar{\mathcal{H}}_4 = \sum_{\boldsymbol{m}=\boldsymbol{n}} c_{\boldsymbol{mn}}\boldsymbol{d}^{\boldsymbol{m}}\boldsymbol{f}^{\boldsymbol{n}} = \sum_{i,j=1}^{3}\omega_{ij}\tau_i\tau_j$$

References

[1] Delp, R. H., and DeBra, D. B., "Rigid Body Attitude Stability and Natural Frequencies in a Circular Orbit," *Journal of Astronautical Sciences*, Vol. 8, 1961, pp. 14–17.

[2] Modi, V. J., and Brereton, R. C., "Periodic Solutions Associated with the Gravity-Gradient-Oriented System Part 1: Analytical and Numerical Determination," *AIAA Journal*, Vol. 7, No. 7, 1969, pp. 1217–1225.

[3] Modi, V. J., and Brereton, R. C., "Periodic Solutions Associated with the Gravity-Gradient-Oriented System Part 2: Stability Analysis," *AIAA Journal*, Vol. 8, No. 8, 1969, pp. 1465–1468.

[4] Beletskii, V. V., *Motion of an Artificial Satellite about Its Center of Mass* (Translated by Z. Lerman), Israel Program for Scientific Translations, Jerusalem, Israel, 1966, pp. 32–75.

[5] Wisdom, J., Peale, S. J., and Mignard, F., "The Chaotic Rotation of Hyperion," *Icarus*, Vol. 58, No. 2, 1984, pp. 137–152.

[6] Likins, P. W., "The Influence on Dynamics and Control Theory of the Spacecraft Attitude Control Problem," 6th Canadian Congress of Applied Mechanics (Vancouver, BC, Canada), 1977, pp. 321–325.

[7] Kane, T. R., Likins, P. W., and Levinson, D. A., *Spacecraft Dynamics*, McGraw–Hill, New York, 1983, pp. 159–211.

[8] Hughes, P. C., *Spacecraft Attitude Dynamics*, Wiley, New York, 1986, pp. 39–138, 281–318.

[9] Marandi, S. R., and Modi, V. J., "Attitude Stability of Rigid Satellites via a Normalized Hamiltonian," *Acta Astronautica*, Vol. 19, No. 4, 1989, pp. 287–299.

[10] Arnold, V. I., "Proof of A. N. Kolmogorov's Theorem on the Conservation of Conditionally Periodic Motions with a Small Variation in the Hamiltonian," *Russian Mathematical Survey*, Vol. 18, No. 5, 1963, pp. 9–36.

[11] Arnold, V. I., *Mathematical Methods of Classical Mechanics*, Springer–Verlag, New York, 1978, pp. 161–300, 381–415.

[12] Arnold, V. I., "Algebraic Unsolvability of the Problem of Lyapunov Stability and the Problem of Topological Classification of Singular Points of an Analytic System of Differential Equations," *Functional Analysis and Its Applications*, Vol. 4, 1970, pp. 173–180.

[13] Tkhai, V. N., "The Stability of Multidimensional Hamiltonian Systems," *PMM* (USSR), Vol. 49, No. 3, 1985, pp. 273–281.

[14] Birkhoff, G. D., *Dynamical Systems*, rev. ed., American Mathematical Society, Providence, RI, 1966, pp. 59–96.

[15] Hearn, A. C., *REDUCE User's Manual, Ver. 3.4*, Rand, Berlin, 1991.

[16] Aygun, M. T., "Attitude Stability of a Rigid Satellite in a Circular Orbit," Ph.D. Thesis, Aeronautical and Astronautical Sciences Faculty, Istanbul Technical Univ., Istanbul, Turkey, June 1995, pp. 48–145.

[17] Arnold, V. I., *Dynamical Systems III*, Springer–Verlag, New York, 1988, pp. 182–199, 251–273.